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Approximation theorems for the iterated Boolean sums of Bernstein operators

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Abstract

For the iterated Boolean sums of Bernstein operators we prove global direct, inverse and saturation results.

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1. Introduction and main result

Over the last 25 years there has been considerable interest in inverse and saturation theorems for the classical Bernstein operators defined by

$$B_n(f, x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k},$$

where $f \in C[0, 1]$ and $x \in [0, 1]$. A key role in this development is played by the important paper of Lorentz (1964) [9]; see also [10, Chapter 7, Section 5] on their saturation classes, and [2] dealing with the corresponding inverse theorems. The results given there are pointwise in nature.

A generalization of Lorentz' saturation theorem [9] and the inverse theorem of Berens and Lorentz [2], as well as the associated global inverse and saturation results for the Bernstein operators, were first given by Zhou [19]. In order to discuss his results, let us first introduce some notation. By $\omega_2(f, \cdot)$, we will denote the second-order modulus of continuity of the

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function $f \in C[0, 1]$. The weighted modulus $\omega_2^*(f, \cdot)$ (essentially introduced in [18], and representing a special case of the main-part modulus defined in [4, p.28]) will be given by

$$\omega_2^*(f, t) := \sup_{0<\eta<t} \sup_{\eta^2<x<1-\eta^2} |\Delta^2_{\eta\sqrt{x(1-x)}} f(x)|.$$

Here Δ^2 denotes a symmetric second-order difference. Using this notation, the following theorem was proved in [19].

Theorem A (Zhou [19]). *Let $\psi(t)$, $0 < t \leqslant 1$, be a nonnegative and nondecreasing function, and let $k > 1$ be a real number such that*

$$t^{3-1/k} \int_t^1 \frac{\psi(\eta)}{\eta^{4-1/k}} \mathrm{d}\eta = \mathrm{O}(\psi(t)).$$

Then the following hold for any $f \in C[0, 1]$.

(i) $|B_n f(x) - f(x)| = \mathrm{O}(\psi((x(1-x)/n)^{1/2}))$ *if and only if* $\omega_2(f, t) = \mathrm{O}(\psi(t))$.

(ii) $\| B_n f - f \| = \mathrm{O}(\psi(n^{-1/2}))$ *if and only if* $\omega_2^*(f, t) = \mathrm{O}(\psi(t))$.

(If not otherwise indicated, in the following $\|\cdot\|$ will always denote the Chebyshev norm, given for $f \in C[0, 1]$ by $\| f \| := \sup\{ |f(t)| : 0 \leqslant t \leqslant 1\}$.)

We also note that the natural L_p variant of $\omega_2^*(f, \cdot)$ was used by the second author [20] to prove inverse theorems for Bernstein–Kantorovich operators. In [19,20], a new technique was introduced which turned out to be most efficient for the proof of inverse and saturation results using a single method; see, e.g., [8], where this method was also used to solve a problem of Heilmann and Müller.

In the present note, we deal with corresponding questions in regard to iterated Boolean sums of Bernstein operators, which have attracted some interest over the last years and which were investigated from a more systematic point of view in [16] (see the references given there for the pertinent historical information). We first introduce some notation.

Let P, Q be operators, $P, Q : X \to X$ for some linear space X. Then the Boolean sum of P and Q is defined to be

$$P \oplus Q := P + Q - PQ.$$

We will be concerned with iterated Boolean sums of the form $B_n \oplus B_n \oplus \cdots \oplus B_n$, and will denote such an M-fold Boolean sum of the Bernstein operators by $\oplus^M B_n$. The easiest way to see that $\oplus^M B_n$ is indeed an approximation operator is to look at the error operator representation

$$I - \oplus^M B_n = (I - B_n)^M,$$

which can be easily verified by introduction. From the last equality one has

$$\oplus^M B_n = I - (I - B_n)^M.$$

The right-hand side of this equality represents a linear combination of powers of a *fixed* Bernstein operator. Such combinations were investigated in the past, but mostly with B_n replaced by other linear operators L_n. The earliest reference in regard to such an approach which we were able to locate is [14]; this method is sometimes attributed to Kharrik (see [13]). In [3, Section 2.2], a proof of Jackson's second theorem is given using this technique. Numerous additional references are listed in [16].

From a numerical point of view, the particular combination given above appears to be of interest, since in the case of discretely defined operators, it uses only the data required by the original operators. In the case of B_n, this is just the set of numbers

$$\left\{f(0), f\left(\frac{1}{n}\right), \ldots, f\left(\frac{n-1}{n}\right), f(1)\right\}.$$

Further computational aspects are discussed in [6].

The operators $\oplus^M B_n$ were introduced independently in [6,11,12]. They were further investigated in [1,5,7,13,16]. See also [15], [17, pp. 36 ff] for a discussion of this approximation method. Several direct, saturation and Voronovskaja-type theorems were given in the papers mentioned. However, the description of the approximation behavior of the operators $\oplus^M B_n$ can still be completed, which is the goal of this note. To be more specific, we will generalize [12, the direct Theorem 4.4] and essentially improve [12, Theorem 4.5] by giving a more elegant version of the saturation statement and by adding the appropriate inverse theorem. We will also obtain the o-saturation class using the special method employed here [12, Corollary 4.2]. A crucial tool for our development will be the so-called Ditzian–Totik modulus $\omega_\phi^{2M}(f, \cdot)$, with $\phi(x) = (x(1-x))^{1/2}$, which is closely related to $\omega_2^*(f, \cdot)$ as defined above. Among others, we shall generalize the global result of [19]. Our theorem demonstrates that the iterated Boolean sums of Bernstein operators quite naturally accelerate the degree of convergence of the underlying "mother operator".

However, for $M > 1$, an analogy of the pointwise saturation result of Lorentz, i.e.,

$$|(I - \oplus^M B_n)(f; x)| = \mathrm{O}\left(\sqrt{\frac{(1-x)x}{n}}\right)^{2M}, \quad \text{iff } \|f^{(2M)}\|_\infty = \mathrm{O}(1),$$

where $\|f\|_\infty := \operatorname{ess\,sup} |f(x)| = \inf\{c \in \mathbb{R}: |f(x)| \leqslant c \text{ a.e.}\}$, seems impossible, as one can show that, e.g., for $f(x) = x^2$,

$$|f(x) - \otimes^M B_n f(x)| = \frac{x(1-x)}{n^M}.$$

Actually we will prove the following theorem.

Theorem 1. *Let $M \geqslant 1$ be fixed. Then, for any $f \in C[0, 1]$; we have*

$$\text{(i)} \qquad \|f - \oplus^M B_n f\| \leqslant C\left\{\omega_\phi^{2M}\left(f, \frac{1}{\sqrt{n}}\right) + \|f\| n^{-M}\right\}.$$

Furthermore, there also holds the Stečkin-type inequality

$$\text{(ii)} \qquad \omega_\phi^{2M}\left(f, \frac{1}{\sqrt{n}}\right) \leqslant \frac{C}{n^{M+1/2}} \sum_{k=1}^{n} k^{M-1/2} \|f - \oplus^M B_k f\|.$$

The o*-saturation class is described as follows*:

$$\text{(iii)} \qquad \| f - \oplus^M B_n f \| = \mathrm{o}\left(\frac{1}{n^M} \right), \quad \textit{if and only if } f \textit{ is a linear function}.$$

Here, $\omega_\phi^{2M}(f, t)$ is defined as in [4, p.10] with $\phi(x) = (x(1-x))^{1/2}$, i.e., for

$$\Delta_{h\varphi}^{2M} f(x) := \sum_{k=0}^{2M} (-1)^k \binom{2M}{k} f(x + (M-k)h\phi(x)), \quad \text{if } x \pm Mh\phi(x) \in [-1, 1],$$

and $\Delta_{h\varphi}^{2M} f(x) := 0$ otherwise, one puts

$$\omega_\phi^{2M}(f, t) := \sup_{0 < h \leqslant t} \| \Delta_{h\varphi}^{2M} f \|.$$

Corollary. *Let* $M \geqslant 1$ *be fixed and* $0 < \alpha \leqslant 2M$ *be given. Then* $\omega_\varphi^{2M}(f, t) = \mathrm{O}(t^\alpha)$ *if and only if* $\| f - \oplus^M B_n f \| = \mathrm{O}(n^{-\alpha/2})$, $n \to \infty$.

Let us finally remark that all constants C figuring in this paper will be independent of f and n. Occasionally a subscript will be added in order to explicitly stress the constant's dependence on a certain quantity.

2. Auxiliary results

Throughout this note $P(D)$ will denote the differential operator

$$P(D) = \phi^2 D^2, \quad \text{with } \phi(x) = \sqrt{x(1-x)}.$$

The following lemma is the central tool for the proof of the theorem.

Lemma 2. *For any polynomial* f, *one has, for* $r = 1, 2, \ldots,$

$$\| P(D)^r f \| \leqslant C(r) \{ \| \phi^{2r} f^{(2r)} \| + \| f \| \} \tag{1}$$

and

$$\| \phi^{2r} f^{(2r)} \| \leqslant C(r) \| P(D)^r f \|. \tag{2}$$

Here $C(r)$ does not depend on f.

Proof. Using induction, one can show that there exist polynomials α_i, β_j such that for $r = 1, 2, \ldots,$ one has

$$P(D)^r = \sum_{i=0}^{r-1} \alpha_i \phi^{2(r-i)} D^{2r-i} + \phi^2 \sum_{i=0}^{r} \beta_i D^i.$$

Now, as

$$\| \phi^{2(r-i)} D^{2r-i} f \| \leqslant C\left[\| \phi^{2r} D^{2r} f \| + \| f \| \right], \quad i = 0, \ldots, r-1, \tag{3}$$

and

$$\| D^i f \| \leqslant C\left[\| \phi^{2r} D^{2r} f \| + \| f \| \right], \quad i = 0, \ldots, r-1,$$

(see [4, p.135]), we see that

$$\left\| \sum_{i=0}^{r-1} \alpha_i \phi^{2(r-i)} D^{2r-i} f + \phi^2 \sum_{i=0}^{r-1} \beta_i D^i f \right\| \leqslant C\left(\| \phi^{2r} D^{2r} f \| + \| f \| \right).$$

On the other hand, for $F = \int_0^x f$, we get by (3) the inequality

$$\begin{aligned} \| \phi^2 f^{(r)} \| = \| \phi^2 F^{(r+1)} \| &\leqslant C\left(\| \phi^{2r} F^{(2r)} \| + \| F \| \right) \leqslant C\left(\| \phi^{2r} f^{(2r-1)} \| + \| f \| \right) \\ &\leqslant C\left(\| \phi^{2r-2} f^{(2r-1)} \| + \| f \| \right) \leqslant C\left(\| \phi^{2r} f^{(2r)} \| + \| f \| \right). \end{aligned}$$

Combining these two inequalities, we obtain

$$\| P(D)^r f \| \leqslant C\left(\| \phi^{2r} f^{(2r)} \| + \| f \| \right),$$

which proves (1).

To prove (2), write $f_0 = f$, and $f_{j+1} = P(D) f_j$, $j = 0, 1, \ldots, r-1$. Putting $\| P(D)^r f \| = \| f_r \| = A$, we have

$$f_{r-1}(x) = \int_{1/2}^{x} \int_{1/2}^{t} \frac{f_r(\eta)}{\phi^2(\eta)} \mathrm{d}\eta \; \mathrm{d}t + ax + b.$$

Obviously,

$$\left\| \int_{1/2}^{x} \int_{1/2}^{t} \frac{f_r(\eta)}{\phi^2(\eta)} \mathrm{d}\eta \; \mathrm{d}t \right\| \leqslant CA.$$

On the other hand, as $f_{r-1}(0) = f_{r-1}(1) = 0$ for $r \geqslant 2$, the last inequality implies that $|a| + |b| \leqslant CA$; hence $\| f_{r-1} \| \leqslant CA$. In general, if $\| f_{j+1} \| \leqslant CA$, then the above argument shows that $\| f_j \| \leqslant CA$, $j \geqslant 1$, with another constant C. Thus we obtain

$$\| f_j \| \leqslant CA, \quad j = 1, 2, \ldots, r. \tag{4}$$

Now, if

$$\left\| \phi^{2i} f_j^{(2i)} \right\| \leqslant CA, \quad j = 0, \ldots, r-i, \quad i = 1, \ldots, k, \tag{5}$$

then, by the definition of f_j, we have

$$\left\| \phi^{2i} \left(P(D) f_j \right)^{(2i)} \right\| \leqslant CA, \quad j = 0, 1, \ldots, r-i-1, \quad i = 1, \ldots, k. \tag{6}$$

Since

$$\left(P(D) f_j \right)^{(2i)} = \left\{ \phi^2 f_j^{(2i+2)} + 2i(\phi^2)' f_j^{(2i+2)} \right\} + i(2i-1)(\phi^2)'' f_j^{(2i)},$$

and since the second term on the right-hand side is bounded by $CA\phi^{-2i}$ due to (5), it follows from (5) and (6) that, for

$$H := \phi^{2i+2} f_j^{(2i+2)} + 2i\phi^{2i}(\phi^2)' f_j^{(2i+1)} = \phi^{2-2i}\left[\frac{\mathrm{d}}{\mathrm{d}x}\phi^{4i} f_j^{(2i+1)}\right],$$

one has $\| H \| \leqslant CA$. On the other hand, using the second expression for H, one gets

$$(\phi^2(x))^{2i} f_j^{(2i+1)}(x) = \int_0^x H(t)(\phi^2(t))^{i-1}\,\mathrm{d}t.$$

Therefore,

$$|(\phi^2(x))^{2i} f_j^{(2i+1)}(x)| \leqslant C\phi^{2i}(x)A, \quad x \in [0, 1].$$

Recalling the definition of H, we see that

$$\| \phi^{2i+2} f_j^{(2i+2)} \| \leqslant CA, \quad j = 0, 1, \ldots, r-i-1, \quad i = 1, \ldots, k.$$

Combining this with (5), we obtain

$$\| \phi^{2i} f_j^{(2i)} \| \leqslant CA, \quad j = 0, 1, \ldots, r-i, \quad i = 1, \ldots, k+1. \tag{7}$$

From (5) and (7), we see that whenever (5) holds for $j = 0, 1, \ldots, r-i$, $i = 1, \ldots, k$, then (7) shows that the same estimate also holds for $j = 0, 1, \ldots, r-i$, $i = 1, \ldots, k+1$. Since for $k = 1$, (5) is (4), it follows from (5) and (7) recursively that

$$\| \phi^{2i} f_j^{(2i)} \| \leqslant CA, \quad j = 0, 1, \ldots, r-i, \quad i = 1, \ldots, r,$$

which implies (2). □

For convenience, we collect some known results in the following lemma.

Lemma 3. *For* $P_m \in \Pi_m$, $m \leqslant \sqrt{n}$,

$$\| \phi^k P_m^{(k)} \| \leqslant C_k m^k \| P_m \|, \quad k = 1, 2, \ldots, \tag{8}$$

$$\| P_m - B_n P_m \| \leqslant \frac{c}{n} \| \phi^2 P_m'' \| \leqslant C \frac{m^2}{n} \| P_m \|, \tag{9}$$

and

$$\left\| P_m - B_n P_m + \frac{1}{2n}\phi^2 P_m^{(2)} \right\| \leqslant C \frac{m^4}{n^2} \| P_m \|. \tag{10}$$

Proof. The proof of (8) can be found in [4, p.91]. Inequalities (9) and (10) can be verified by using (8) and Taylor's formula. □

In the following lemma we generalize inequality (10) of Lemma 3 by proving a Voronovskaja-type theorem for certain polynomial spaces. Related results can be found in [5], [7, Theorem 1], [15,17]. The form given below is exactly what we need for our purposes.

Lemma 4. *Let $P_m \in \Pi_m$, $m \leqslant \sqrt{n}$, $r = 1, 2, \ldots$. Then,*

$$\left\| (I - B_n)^r P_m - \left(\frac{-1}{2n} \right)^r P(D)^r P_m \right\| \leqslant C_r \frac{m^{2r+2}}{n^{r+1}} \| P_m \|,$$

where C_r does not depend on m, n or P_m.

Proof. First we point out that for $P_m \in \Pi_m$,

$$\| (I - B_n)^k P_m \| \leqslant C_k \frac{m^{2k}}{n^k} \| P_m \|. \tag{11}$$

To see this, first note that for $P_m \in \Pi_m$,

$$(I - B_n) P_m \in \Pi_m.$$

Now, by Lemma 3, Eq. (9).

$$\| (I - B_n)^k P_m \| = \| (I - B_n)(I - B_n)^{k-1} P_m \| \leqslant C \frac{m^2}{n} \| (I - B_n)^{k-1} P_m \|.$$

Iterated use of (9) then yields

$$| (I - B_n)^k P_m \| \leqslant C_k \frac{m^{2k}}{n^k} \| P_m \|.$$

To complete the proof of this lemma, we define

$$L^j = (2n)^{-j} (I - B_n)^{r-j} P(D)^j, \quad 0 \leqslant j \leqslant r.$$

Note first that

$$L^0 - (-1)^r L^r = \sum_{j=0}^{r-1} (-1)^j (L^j + L^{j+1}).$$

But,

$$L^0 - (-1)^r L^r = (I - B_n)^r - \left(\frac{-1}{2n} \right)^r P(D)^r.$$

Thus, by (10) and (11),

$$\begin{aligned} &\left\| (I - B_n)^r P_m - \left(\frac{-1}{2n} \right)^r P(D)^r P_m \right\| \\ &\quad \leqslant \sum_{j=0}^{r-1} \| (L^j + L^{j+1}) P_m \| \\ &\quad \leqslant \sum_{j=0}^{r-1} \frac{1}{(2n)^j} \left\| (I - B_n)^{r-j-1} \left\{ (I - B_n) P(D)^j P_m + \frac{1}{2n} P(D)^{j+1} P_m \right\} \right\| \end{aligned}$$

$$\leqslant \sum_{j=0}^{r-1} C_j \frac{1}{n^j} \frac{m^{2(r-j-1)}}{n^{r-j-1}} \left\| (I-B_n)P(D)^j P_m + \frac{1}{2n} P(D)^{j+1} P_m \right\|$$

$$\leqslant C \frac{m^{2r+2}}{n^{r+1}} \sum_{j=0}^{r-1} m^{-2j} \| P(D)^j P_m \|.$$

Now, (1) and (8) imply

$$\sum_{j=0}^{r-1} m^{-2j} \| P(D)^j P_m \| \leqslant C \| P_m \|. \quad \square$$

3. Proof of Theorem 1

We first recall the following facts (see [4, pp. 79, 84]). Let $P_m \in \Pi_m$ be such that $\| f - P_m \| = E_m(f)$. Then,

$$E_m(f) \leqslant C_k \omega_\phi^k \left(f, \frac{1}{m} \right), \quad k = 1, 2, \ldots, \quad m > k, \tag{12}$$

and

$$\| \phi^k P_m^{(k)} \| \leqslant C_k m^k \omega_\phi^k \left(f, \frac{1}{m} \right), \quad k = 1, 2, \ldots. \tag{13}$$

Furthermore, let $\| P_{2^i} - f \| = E_{2^i}(f)$, $i = 0, 1, \ldots$. Then, if $2^N \leqslant \sqrt{n}$, write

$$P_{2^N} = \sum_{i=1}^{N} (P_2^i - P_2^{i-1}) + P_1.$$

Using Lemma 4 for each term in the above sum, we get for $k = 1, 2, \ldots,$

$$\left\| (I-B_n)^k P_{2^N} - \left(\frac{-1}{2n} \right)^k P(D)^k P_{2^N} \right\| \leqslant \frac{C}{n^{k+1}} \sum_{i=0}^{N} (2^i)^{2k+2} E_{2^i}(f). \tag{14}$$

In the following, we assume that $n = m^2$ and $m = 2^N$ for convenience, and now proceed to prove the theorem.

Proof of (i). Let $P_m \in \Pi_m$ be such that $\| f - P_m \| = E_m(f)$. Then we have

$$\| f - \oplus^M B_n f \| \leqslant \| P_m - \oplus^M B_n P_m \| + C E_m(f)$$

$$\leqslant \left\| (I-B_n)^M P_m - \left(\frac{-1}{2n} \right)^M P(D)^M P_m \right\| + (2n)^{-M} \| P(D)^M P_m \| + C E_m(f).$$

Using (14) with $k = M$ for the first term of the right-hand side, (1) and (13) with $k = 2M$ for the second one, and (12) for the last one, we arrive at

$$\| f - \oplus^M B_n f \| \leqslant C \left\{ \omega_\phi^{2M} \left(f, \frac{1}{\sqrt{n}} \right) + \frac{1}{n^M} \| f \| + \frac{1}{n^{M+1}} \sum_{i=0}^{N} (2^i)^{2M+2} E_{2^i}(f) \right\}.$$

Again using (12), we get

$$\frac{1}{n^{M+1}} \sum_{i=1}^{N} (2^i)^{2M+2} E_{2^i}(f) \leqslant \frac{C}{n^{M+1}} \sum_{i=1}^{N} (2^i)^{2M+2} \frac{n^M}{(2^i)^{2M}} \omega_\phi^{2M}\left(f, \frac{1}{\sqrt{n}}\right) + \frac{C}{n^{M+1}} \| f \|$$
$$\leqslant C\omega_\phi^{2M}\left(f, \frac{1}{\sqrt{n}}\right) + \frac{C}{n^{M+1}} \| f \|.$$

Combining the two latter estimates, we get (i).

To prove (ii), first note that (see [4, pp. 125, 156])

$$\left\| \phi^{2M+2}(B_k f)^{(2M+2)} \right\| \leqslant \begin{cases} Ck^{M+1}\| f \|, & f \in C[0, 1], \\ C\| \phi^{2M+2} f^{(2M+2)} \|, & f^{(2M+2)} \in C[0, 1]. \end{cases}$$

Thus, as $\oplus^{M+1} B_k f$ is a linear combination of B_k^j, $j = 1, 2, \ldots, M+1$, the estimate above also holds for $\oplus^{M+1} B_k f$. Therefore, by the equivalence of the K-functional and the modulus ω_ϕ, we have, for any $g^{(2M-2)} \in C[0, 1]$,

$$\omega_\phi^{2M+2}(f, t) \leqslant C\left\{ \| f - \oplus^{M+1} B_k f \| + t^{2M+2} \left\| \phi^{2M+2}\left(\oplus^{M+1} B_k f\right)^{(2M+2)} \right\| \right\}$$
$$\leqslant C\left\{ \| f - \oplus^{M} B_k f \| + t^{2M+2} k^{M+1} \left\{ \| f - g \| + k^{-M-1} \| \phi^{2M+2} g^{(2M+2)} \| \right\} \right\}.$$

This implies

$$\omega_\phi^{2M+2}(f, t) \leqslant C\left\{ \| f - \oplus^{M} B_k f \| + t^{2M+2} k^{M+1} \omega_\phi^{2M+2}\left(f, \frac{1}{\sqrt{k}}\right) \right\}.$$

Hence (see, e.g. [4, p.123, implication 9.3.12 $\Rightarrow$ 9.3.13]),

$$\omega_\phi^{2M+2}\left(f, \frac{1}{\sqrt{n}}\right) \leqslant Cn^{-M-1/2}\left(\sum_{k=1}^{n} k^{M-1/2} \| f - \oplus^{M} B_k f \| + \| f \| \right). \tag{15}$$

On the other hand, the definition of the K-functional also shows that, for $P_m \in \Pi_m$ with $\| f - P_m \| = E_m(f)$,

$$\omega_\phi^{2M}\left(f, \frac{1}{\sqrt{n}}\right) \leqslant C\left\{ \| f - P_m \| + \frac{1}{n^M} \| \phi^{2M} P_m^{(2M)} \| \right\}.$$

Now, by (2),

$$\frac{1}{n^M} \| \phi^{2M} P_m^{(2M)} \| \leqslant C\left\{ \| P_m - \oplus^{M} B_n P_m \| + \left\| P_m - \oplus^{M} B_n P_m - \left(\frac{-1}{2n}\right)^M P(D)^M P_m \right\| \right\},$$

so that

$$\omega_\phi^{2M}\left(f, \frac{1}{\sqrt{n}}\right) \leqslant C\left\{ \| f - P_m \| + \| f - \oplus^{M} B_n f \| + \left\| P_m - \oplus^{M} B_n P_m - \left(\frac{-1}{2n}\right)^M P(D)^M P_m \right\| \right\}.$$

Hence, it follows from (12), (14) and (15) that

$$\omega_\phi^{2M}\left(f, \frac{1}{\sqrt{n}}\right) \leqslant C\left\{\| f - \oplus^M B_n f \| + n^{-M-1/2}\left(\sum_{k=1}^{n} k^{M-1/2} \| f - \oplus^M B_k f \| + \| f \| \right) + \frac{1}{n^{M+1}} \sum_{i=1}^{N} (2^i)^{2M+2} \omega_\phi^{2M+2}(f, 2^{-i}) \right\}$$

$$\leqslant C\left\{\| f - \oplus^M B_n f \| + n^{-M-1/2}\left(\sum_{k=1}^{n} k^{M-1/2} \| f - \oplus^M B_k f \| + \| f \| \right)\right\}.$$

Multiplying by $n^{M-1/2}$ on both sides of the above inequality and taking the sum from n to $2n$, we obtain, by the monotonicity of $\omega_\phi^{2M}(f, t)$,

$$n^{M+1/2}\omega_\phi^{2M}\left(f, \frac{1}{\sqrt{2n}}\right) \leqslant C\left\{ \sum_{k=1}^{2n} k^{M-1/2} \| f - \oplus^M B_k f \| + \| f \| \right\}. \tag{16}$$

Finally, since for any linear function l, $\omega_\phi^{2M}(f-l, t) = \omega_\phi^{2M}(f, t)$,

$$\|(f-l) - \oplus^M B_n(f-l)\| = \| f - \oplus^M B_n f \| \quad \text{and} \quad \| f - \oplus^M B_1 f \| \geqslant E_1(f),$$

(ii) follows from (16).

To prove (iii), we notice that, since for any linear function f, $\| f - \oplus^M B_n f \| = 0$, we need only verify that

$$\| f - \oplus^M B_n f \| = o(n^{-M}) \tag{17}$$

implies that f is a linear function. To see this, we have by (ii) that (17) implies

$$\omega_\phi^{2M}(f, t) = o(t^{2M}).$$

Thus, $f \in \Pi_{2M-1}$. Taking $P_m = f$ in Lemma 4 and multiplying by n^r (now $r = 2M$) on both sides of the inequality in Lemma 4, and finally letting $n \to \infty$, we obtain, due to (17), that $P(D)^M f = 0$. Denote $P(D)^{M-1} f$ by H. As $\phi^2 H'' = P(D)H = 0$, H is a linear function, i.e., $H = ax + b$ for some a and b. But, by the definition of H, if $M - 1 \geqslant 1$, $H(0) = H(1) = 0$. Hence $P(D)^{M-1} f = 0$. In this way, we get $P(D)^j f = 0$, $j = 1, 2, \ldots, M$. Therefore, $P(D) f = 0$, which implies that f is a linear function. □

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References

[1] P.N. Agrawal and H.S. Kasana, On the iterative combinations of Bernstein polynomials, *Demonstratio Math.* **17** (3) (1984) 777–783.

[2] J. Berens and G.G. Lorentz, Inverse theorems for Bernstein polynomials, *Indiana Univ. Math. J.* **21** (1972) 693–708.
[3] P.L. Butzer and R.J. Nessel, *Fourier Analysis and Approximation* (Academic Press, New York, 1971).
[4] Z. Ditzian and V. Totik, *Moduli of Smoothness* (Springer, New York, 1987).
[5] M.Š. Džamalov, On a theorem of E.V. Vornovskoĭ, in: *Operators and their Applications / Approximation of Functions / Equalities* (Leningrad, 1985, in Russian) 22–27.
[6] G. Felbecker, Linearkombinationen von iterierten Bernsteinoperatoren, *Manuscripta Math.*, **29** (1979) 229–246.
[7] W. Gawronski and U. Stadtmüller, Linear combinations of iterated generalized Bernstein functions with an application to density estimation, *Acta Sci. Math. (Szeged)* **47** (1984) 205–221.
[8] H.H. Gonska and X.-l. Zhou, A global inverse theorem on simultaneous approximation by Bernstein–Durrmeyer operators, *J. Approx. Theory* **67** (1991) 284–302.
[9] G.G. Lorentz, Inequalities and saturation classes of Bernstein polynomials, in: P.L. Butzer and J. Korevaar, Eds., *On Approximation Theory*, Proc. Conf. Math. Res. Inst. Oberwolfach 1963 (Birkhäuser, Basel, 1964) 200–207.
[10] G.G. Lorentz, *Approximation of Functions* (Chelsea, New York, 2nd ed., 1986).
[11] G. Mastroianni and M.R. Occorsio, Una generalizzazione dell'operatore di Bernstein, *Rend. Accad. Sci. Fis. Mat. Napoli* **44** (1977) 151–169.
[12] C. Micchelli, The saturation class and iterates of the Bernstein polynomials, *J. Approx. Theory* **8** (1973) 1–18.
[13] G.I. Natanson, Application of the method of I.P. Natanson and I.Yu. Kharrik in the algebraic case, in: *Operator Theory and Function Theory, No. 1* (Leningrad Univ., Leningrad, 1983, in Russian) 166–170.
[14] I.P. Natanson, On the approximation of multiply differentiable periodic functions by means of singular integrals, *Dokl. Akad. Nauk SSSR* **82** (1952) 337–339 (in Russian).
[15] T.P. Pendina, Iterations of positive linear operators of exponential type and of Landau polynomials, in: *Geometric Problems of the Theory of Functions and Sets* (Kalinin, Gos. Univ., Kalinin, 1987, in Russian) 105–111.
[16] J. Sevy, Acceleration of convergence of sequences of simultaneous approximants, Ph.D. Thesis, Drexel Univ. Philadelphia, PA, 1991.
[17] V.S. Videnskiĭ, Bernstein polynomials, "A.I. Gerzen" State Pedagogical Inst., Leningrad, 1990 (in Russian).
[18] X.-l. Zhou, Approximation of integrable functions by Baskakov-type operators, Master Thesis, Hangzhou Univ., Hangzhou, 1981 (in Chinese).
[19] X.-l. Zhou, On Bernstein polynomials, *Acta Math. Sinica* **28** (6) (1985) 848–855 (in Chinese).
[20] X.-l. Zhou, On the approximation degree of Bernstein–Kantorovich polynomials in $L_p(0, 1)$, *Adv. in Math. (China)* **14** (2) (1985) 147–157 (in Chinese).